Bipartite matching with random arrival order

Note Title 2/7/2013

- Bipartite graph (L, R, E).
- Vertices in R arrive in a random order.
- Find max-size matching

Algorithm : RANKING: pick a permutation
$\sigma \in S_n$ u.a.r. $\sigma : [n] \to L$
$\sigma(\lambda)$ = vertex in L with rank λ
Given j, match it to $\arg\min_{i : i \sim j} \{\sigma(i) : i \text{ not yet matched}\}$

Denote by $\pi(r)$ the vertex in R that arrives r^{th} in the order, that is, of rank r. $\pi : [n] \to R$

- Recall, without randomization on the R side, we get $1 - 1/e$.
- For the $\vec{B}$-matching problem with large B_j's, we get $1-\varepsilon$, $\varepsilon \to 0$ as $B \to \infty$.
- Better than $1-1/e$?

Techniques we have seen:

- Primal - Dual : gives only $1-1/e$.
  Is there a way to push to get better?
  May be !! But don't know.

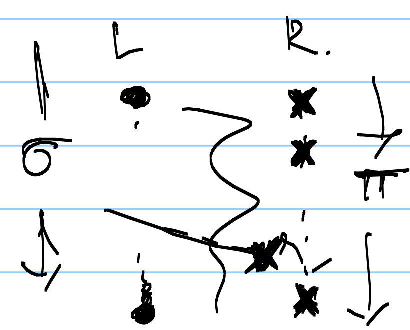
- Hybrid Algorithm: Also only $1-1/e$ for matching.
  Need bigger capacity for better CR.
  Only for iid, not random permutation.

- Sample + Learn: only for bigger capacities.
  Not enough time to learn.

New Technique: Factor-revealing LP.
- solution gives a CR.
- Solve using a computer.
- Use certain (simple) properties of the algo.

Observation :-

Can switch "online" & "offline" sides.

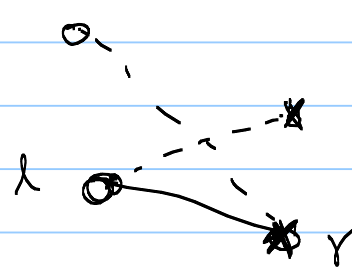


Notation :-

$$x(\sigma, \pi, l, r) = \begin{cases} 1 & \text{if } \sigma(l) \text{ is matched to } \pi(r) \\ 0 & \text{otherwise} \end{cases}$$

Dominance :- If $\sigma(l) \nsim \pi(r)$, then either $\sigma(l)$ is matched to $\pi(r')$ with $r' \leq r$, or $\pi(r) \xrightarrow{M} \sigma(l')$ with $l' \leq l$.

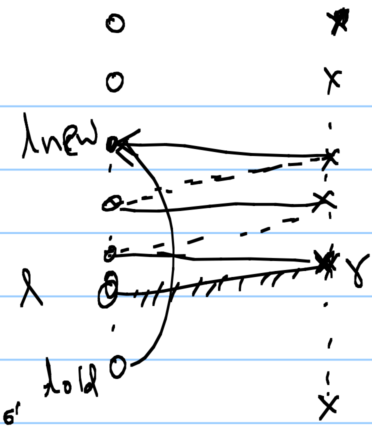
M = matching found by Algo.



$$(1) - \sum_{l'=1}^{l} x(\sigma, \pi, l', r) + \sum_{r'=1}^{r} x(\sigma, \pi, l, r') \geq 1.$$

— cont'd. . .

Monotonicity :-



$\sigma' =$ move l_{old} to l_{new} in σ.

(1) If $l \leq l_{new}$,
Then $r \xrightarrow{M} l$ in $\sigma \Leftrightarrow$ in σ'
i.e. $x(\sigma,\pi,l,r) = x(\sigma',\pi,l,r) \quad \forall\, l < l_{new}$

(2) If $l_{new} \leq l < l_{old}$, then

$$\sum_{l'=l_{new}}^{l+1} x(\sigma',\pi,l',r) \geq \sum_{l'=l_{new}}^{l} x(\sigma,\pi,l,r)$$

also $\sum_{l'=1}^{l+1}$ —"— $\geq \sum_{l'=1}^{l}$ —"—

Sim^{ly}, for moving a vertex from $r_{old} \to r_{new}$,
Let π' be the new permutation.

(3) If $r < r_{new}$, then

$$x(\sigma,\pi',l,r) = x(\sigma,\pi,l,r)$$

If $r_{new} \leq r < r_{old}$, then.

$$\sum_{r'=r_{new}}^{r+1} x(\sigma,\pi',l,r') \geq \sum_{r'=r_{new}}^{r} x(\sigma,\pi,l,r)$$

———————— × ————————

Suppose $OPT = K$

$|L| = |R| = n$.

Factor-revealing LP:

$$\text{minimize} \quad \frac{1}{k(n!)^2} \sum_{\sigma,\pi,l,r} x(\sigma,\pi,l,r)$$

s.t. (1), (2) & (3).

$$x(\sigma,\pi,l,r) \geq 0.$$

Lemma:- OPT value of LP ≤ CR of Ranking

Proof:- Ranking gives a feasible soln to the LP.
Objective fn. = CR.

∴ Min. can only be smaller. ∎

Take inf over all n.

- LP is too big ☹. ∴ Need smaller LP. ☺

- Get rid of σ & π. Take expectations!

Dominance with ~~expectations~~ Probabilities:- $\forall\, l, r$

$$\Pr_r[\pi(r) \xrightarrow{M} \sigma(l') : l' \leq l] + \Pr_r[\sigma(l) \xrightarrow{M} \pi(r') : r' \leq r] \geq \Pr_r[\sigma(l) \sim \pi(r)] \geq \frac{k}{n^2}$$

∵ all we know is that there are k edges, since OPT = k.

$$\Pr_r[\pi(r) \xrightarrow{M} \sigma(l') : l' \leq l \;\&\; \pi(r) \xrightarrow{OPT} \sigma(l)] + \Pr[\sigma(l) \xrightarrow{M} \pi(r') : r' \leq r \;\&\; \sigma(l) \xrightarrow{OPT} \pi(r)] \geq \frac{k}{n^2}$$

Monotonicity with probabilities:

Fix $l_{old} = n$. $\forall\, l, l_{new} < n,\ r.\ \forall p$

$\sigma' =$ pick a random perm. Move last element to position l_{new}.

$\equiv$ a random permutation.

If $l \leq l_{new}$.

$$\Pr\left[\sigma(l) \xrightarrow{M} \pi(r)\ \&\ \sigma(p) \xrightarrow{OPT} \pi(r)\right] = \Pr\left[\sigma(l) \xrightarrow{M} \pi(r)\ \&\ \sigma(p^{new}) \xrightarrow{OPT} \pi(r)\right]$$

$$p^{new} = \begin{cases} p & \text{if } p < l_{new} \\ p+1 & \text{if } l_{new} \leq p < n \\ l_{new} & \text{if } p = n \end{cases}$$

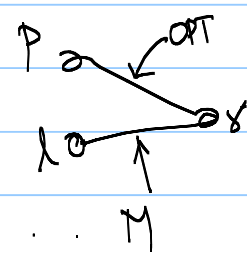
If $l_{new} \leq l < n$

$$\Pr\left[\sigma(l') \xrightarrow{M} \pi(r),\ l' \leq l+1\ \&\ \sigma(p^{new}) \xrightarrow{OPT} \pi(r)\right]$$

$$\geq \Pr\left[\sigma(l') \xrightarrow{M} \pi(r),\ l' \leq l,\ \&\ \sigma(p) \xrightarrow{OPT} \pi(r)\right]$$

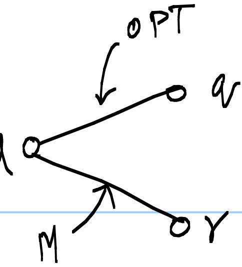
Note:- Probabilities involve where $\pi(r)$ is matched to in M & OPT. $\therefore$ All of these can be written in terms of:

$x_1(l, r, p) =$ Pr. ~~that~~



only n^3 variables.

IIIly, we also have

$x_2(r, l, q) = \Pr$ that 

We also have

$$\begin{array}{ccc} \Pr[\sigma(l) \xrightarrow{M} \pi(r)] & = & \Pr[\sigma(l) \xrightarrow{M} \pi(r)] \\ \| & & \| \\ \sum_p x_1(l, r, p) & & \sum_q x_2(r, l, q) \end{array}$$

which (divided by k) is also the objective.

Simplifications :-

1. By symmetry, $x_1(l, r, p) = x_2(l, r, p)$
2. w.l.o.g $k = n$.
3. Remove some redundant constraints.

This is a poly-size FR-LP, one for every n.
Gives a lower bound on CR, for that n.

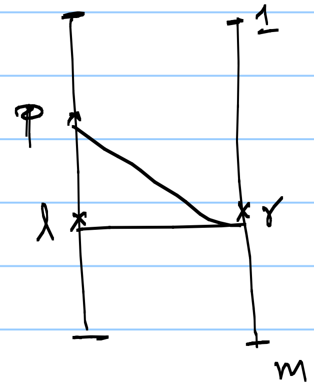
$\therefore$ No way to get a lower bound $\forall n$!

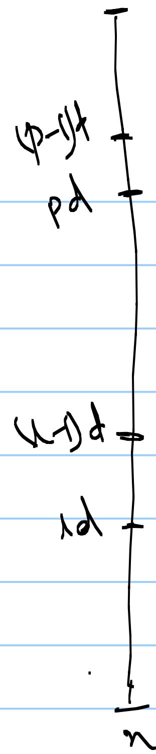
Project down: Given a solution to LP(n)
get a soln. to LP$'$(m), $m = n/d$. s.t.

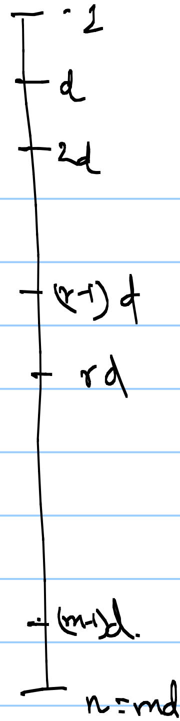
$$LP'(m) \leq LP(n)$$

bd. on CR. $\therefore$ opt of $LP'(m) \leq \inf_{n=md} LP(n)$

$l, r, p \in [m]$







$\mathbb{P}_r$

Dominance:-

$$\Pr_{r}\left[\begin{array}{l}\pi(r) \xrightarrow{M} \sigma(1..\lambda)\\ \& \ \pi(r) \xrightarrow{OPT} \sigma(\lambda)\end{array}\right] + \Pr_{r}\left[\begin{array}{l}\sigma(\lambda) \xrightarrow{M} \pi(1..r)\\ \& \ \sigma(\lambda) \xrightarrow{OPT} \pi(p)\end{array}\right]$$

$$\geq \frac{1}{n}.$$

$CR = 0.696.$